1. Details of Module and its structure

Module Detail	
Subject Name	Physics
Course Name	Physics 01 (Physics Part-1, Class XI)
Module Name/Title	Unit 5, Module 2, System of Particles Chapter 7, System of Particles and Rotational Motion
Module Id	Keph_10702_eContent
Pre-requisites	Knowledge of Kinematics, laws of motion, basic vector algebra
Objectives	After going through this lesson, the learners will be able to: • Understand the significance of velocity and acceleration of centre of mass • Relate momentum of system of particle with velocity of centre of mass • Relate net external force on a system of particles and acceleration of the centre of mass • Calculate the Kinetic energy of a system of particles • Apply Work energy theorem for a system of particles
Keywords	• Rigid body rotation, centre of mass Kinetic energy of a system of particles acceleration of the centre of mass

2. Development Team

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1. UNIT SYLLABUS

Unit V: Motion of System of Particles and Rigid body

Chapter 7: System of particles and Rotational Motion

Centre of mass of a two-particle system; momentum conservation and centre of mass motion. Centre of mass of a rigid body; Centre of mass of a uniform rod.

Moment of a force; torque; angular momentum; law of conservation of angular momentum and its applications.

Equilibrium of rigid bodies; rigid body rotation and equations of rotational motion; comparison of linear and rotational motions.

Moment of inertia; radius of gyration; values of moments of inertia for simple geometrical objects (no derivation). Statement of parallel and perpendicular axes theorems and their applications.

2. MODULE-WISE DISTRIBUTION OF UNIT SYLLABUS 8 Modules

The above unit has been divided into 8 modules for better understanding.

Module 1	• Rigid body • Centre of mass • Distribution of mass • Types of motion: Translatory, circulatory and rotatory
Module 2	• Centre of mass • Application of centre of mass to describe motion • Motion of centre of mass
Module 3	• Analogy of circular motion of a point particle about a point and different points on a rigid body about an axis • Relation v = rω • Kinematics of rotational motion
Module 4	• Moment of inertia • Difference between mass and moment of inertia • Derivation of value of moment of inertia for a lamina about a vertical axis perpendicular to the plane of the lamina • S I Unit • Radius of gyration • Perpendicular and Parallel axis theorems
Module 5	• Torque • Types of torque • Dynamics of rotational motion • $\tau = I\alpha$
Module 6	• Equilibrium of rigid bodies • Condition of net force and net torque • Applications
Module 7	• Law of conservation of angular momentum and its applications • Applications
Module 8	• Rolling on plane surface • Horizontal • Inclined surface • Applications

Module 2

3. WORDS YOU MUST KNOW

- Rigid body: An object for which individual particles continue to be at the same separation over a period of time.

- Point object: If the position of an object changes by distances much larger than the dimensions of the body the body may be treated as a point object.

- Frame of reference: Any reference frame the coordinates(x,y,z), which indicate the change in position of object with time. Inertial frame of reference is stationary or moving with a constant velocity. Non inertial frames of reference are accelerating rotating frames.
- Observer: Someone who is observing objects from any frame, from inertial frames such an observer is stationary with respect to the surrounding or is in uniform motion.
- Rest: A body is said to be at rest if it does not change its position with respect to its surrounding - in time.
- Motion: A body is said to be in motion if it changes its position with respect to its surroundings.
- Time elapsed: Time interval between any two observations of an object.
- Motion in one dimension: When the position of an object can be shown by change in any one coordinate out of the three (x, y, z), also called motion in a straight line.
- Motion in two dimension: When the position of an object can be shown by changes any two coordinate out of the three (x, y, z), also called motion in a plane.
- Motion in three dimension: When the position of an object can be shown by changes in all three coordinate out of the three (x, y, z).

- Distance: The path length an object has moved from its starting position to reach a final position. Its SI unit is m, it is a scalar quantity. This can be zero or positive.

- Displacement: The distance an object has moved from its starting position moves in a particular direction. It is the shortest path length between an initial and a final position. SI unit: m, it is a vector quantity. This can be zero, positive or negative.

- Position vector: A vector representing the location of a point in space with respect to a fixed frame of reference

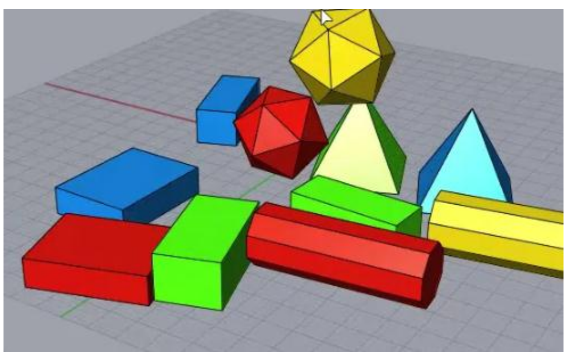
- Force: A push or a pull that can change the state of rest or motion of a body. It can also deform a body.

- Center of mass: The centre of mass of a system of particles moves as if all the mass of the system was concentrated at this centre of mass and all external forces were applied at that point.

4. INTRODUCTION

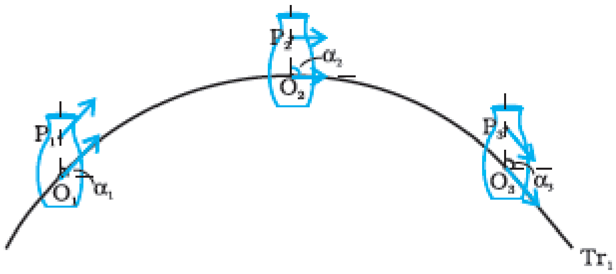
We have been considering rigid bodies.

A *rigid body* is an object of finite extent in which all the distances between the component particles are constant. No truly rigid body exists; external forces can deform any solid. For our purposes, then, a rigid body is a solid which requires large forces to deform it appreciably.

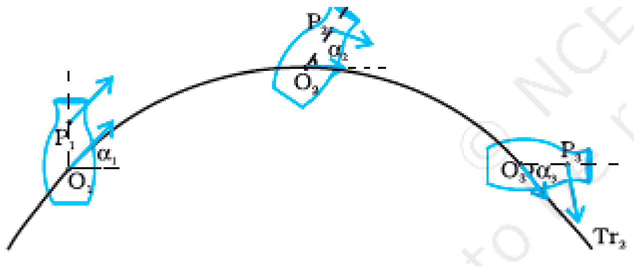


A rigid body is a body with a perfectly definite and unchanging shape. The distances between all pairs of particles of such a body do not change.

A **change in the position of a particle** in three-dimensional space can be completely specified by three coordinates.



Motion of a rigid body which is pure translation



Motion of a rigid body which is a combination of translation and rotation

A change in the position of a rigid body is more complicated to describe. It can be regarded as a combination of two distinct types of motion: translational motion and rotational motion.

Purely **translational motion** occurs when every particle of the body has the same instantaneous velocity as every other particle; then the path traced out by any particle is exactly parallel to the path traced out by every other particle in the body. Under translational motion, the change in the position of a rigid body is specified completely by three coordinates such as *x*, *y*, and *z* giving **the displacement** of any point, such as the center of mass, fixed to the rigid body.

Purely **rotational motion occurs if every particle in the body moves in a circle about a single line. This line is called the axis of rotation. The axis of rotation need not go through the body**

The **examples taken up in the module 1 of this unit help us to infer that the centre of mass of a system of particles moves as if all the mass of the system was concentrated at the centre of mass and all the external forces were applied at that point.**

Also, for finding the motion of the centre of mass no knowledge of internal forces of the system of particles is required; for this purpose we need to know only the external forces.

NEED TO DESCRIBE CENTRE OF MASS

The rigid body may move in different ways; example: move along a straight or curved line, it may move in a circle, rotate or vibrate.

The rigid body is a system may be a collection of particles in which there may be all kinds of internal motions, or it may be a rigid body which has either pure translational motion or combination of translational and rotational motion. Whatever is the system and the motion of its individual particles, the position vector, velocity and acceleration of the centre of mass can be given according to following equations

Position vector of centre of mass $\overrightarrow{\mathbf{r}_{CM}} = \frac{\sum_{i=1}^{n} \mathbf{m_i}\vec{\mathbf{r_i}}}{\sum_{i=1}^{n} \mathbf{m_i}}$

Velocity of centre of mass $\overrightarrow{\mathbf{v}_{CM}} = \frac{\sum_{i=1}^{n} \mathbf{m_i}\vec{\mathbf{v_i}}}{\sum_{i=1}^{n} \mathbf{m_i}}$

Acceleration of centre of mass $\overrightarrow{\mathbf{a}_{CM}} = \frac{\sum_{i=1}^{n} \mathbf{m_i}\vec{\mathbf{a_i}}}{\sum_{i=1}^{n} \mathbf{m_i}}$

Therefore instead of treating extended bodies as single particles, as we have done in earlier chapters of kinematics and laws of motion, we can now treat them as systems of particles.

In this module, we will consider motion of center of mass

5. ACCELERATION OF CENTRE OF MASS AND FORCE ON A SYSTEM OF PARTICLES

The translational component of their motion, i.e. the motion centre of mass of the system, by considering the mass of the whole system to be concentrated at the centre of mass and all the external forces on the system to be acting at the centre of mass. The fact that external forces can only affect the motion of the centre of mass can help us in describing and separate the translational motion of a rigid body or a system of particles, which may be rotating as well, with all kinds of internal motion.

From equation (3) we can write

$$\sum_{i=1}^{n} \mathrm{m_i}\, \overrightarrow{\mathrm{a_{CM}}} = \sum_{i=1}^{n} \mathrm{m_i} \overrightarrow{\mathrm{a_i}} = \sum_{i=1}^{n} \overrightarrow{\mathrm{F_i}}$$

As $\sum_{i=1}^{n} \overrightarrow{\mathrm{F_i}} = \overrightarrow{\mathrm{F_{ext}}}$ = sum of all the forces acting on the given system

$$\text{and} \sum_{i=1}^{n} \mathrm{m_i} = \mathrm{M} = \text{mass of the system}$$

Therefore $$\overrightarrow{\mathrm{F_{ext}}} = \mathrm{M}\, \overrightarrow{\mathrm{a_{CM}}}$$

The above equation is the **Newton's second law for a system of particles.**

The centre of mass accelerates as if it was a point particle of mass M and net external force were applied at this point.

In fact, it does not matter the external force is really applied at the centre of mass.

Therefore one can determine the acceleration of the centre of mass of a given system of particles as

$$\overrightarrow{\mathbf{a_{CM}}} = \frac{\overrightarrow{\boldsymbol{F}}_{\boldsymbol{ext}}}{\mathbf{M}}$$

i.e. the net external force acting on the system of particles per unit net mass of the system.

APPLICATION OF THE IDEA OF CENTER OF MASS

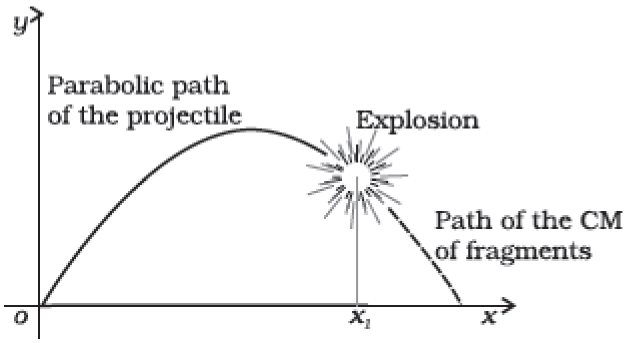
EXAMPLE

Consider a **Diwali Rocket cracker,** following the usual parabolic trajectory, explodes into fragments midway in air. The forces leading to the explosion are **internal forces**. They contribute nothing to the motion of the centre of mass.

The internal forces can contribute to the velocity and or acceleration to the fragments of the projectile.

Therefore**, the fragments of projectile will move in different directions such that the centre of mass of the system remains unaffected**.

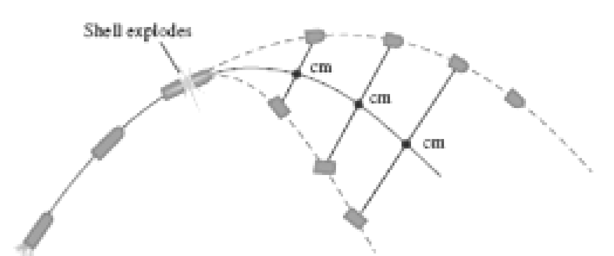
The total external force, namely, the force of gravity acting on the body, is the same before and after the explosion. The centre of mass under the influence of the external force continues, therefore, along the same parabolic trajectory as it would have followed if there were no explosion



The centre of mass of the fragments of the projectile continues along the same parabolic path which it would have followed if there were no explosion.

In the figure below rocket shell traveling in a parabolic trajectory (neglecting air resistance) explodes in flight, splitting into two fragments with equal mass. The fragments follow new parabolic paths, but the center of mass continues on the original parabolic trajectory, just as though all the mass were still concentrated at that point.

NOTICE: The fragments are not travelling along the trajectory of the center of mass



6. VELOCITY OF CENTRE OF MASS AND MOMENTUM OF A SYSTEM OF PARTICLES

Let us consider a system of n particles with massesm$_1$, m$_2$...m$_n$ respectively and velocities$\vec{v_1}$, $\vec{v_2}$,....... $\vec{v_n}$ respectively.

We can write the expression of velocity of the centre of mass of a system of particles in terms of the velocity of the constituent particles as

$$\overrightarrow{v_{CM}} = \frac{\sum_{i=1}^{n} m_i \vec{v_i}}{\sum_{i=1}^{n} m_i}$$

As mass of the system can be written as $\sum_{i=1}^{n} m_i = M$

Hence we can write

$$\overrightarrow{v_{CM}} = \frac{\sum_{i=1}^{n} m_i \vec{v_i}}{M}$$

$$M\,\overrightarrow{v_{CM}} = \sum_{i=1}^{n} m_i \vec{v_i}$$

$$M\,\overrightarrow{v_{CM}} = m_1\vec{v_1} + m_2\vec{v_2} + \cdots \ldots + m_n\vec{v_n}$$

$$M\,\overrightarrow{v_{CM}} = \vec{p_1} + \vec{p_2} + \cdots \ldots + \vec{p_n} = \sum_{i=1}^{n} \vec{p_i}$$

$$M\,\overrightarrow{v_{CM}} = \sum_{i=1}^{n} \vec{p_i}$$

$$\mathbf{M\,\overrightarrow{v_{CM}} = \text{vector sum of linear momentums of individual particles of system}}$$

Thus, the total momentum of a system of particles is equal to the product of the total mass of the system and the velocity of its centre of mass.

One can also write the velocity of centre of mass a given system of particles as

$$\overrightarrow{\mathbf{v}_{\mathbf{CM}}} = \frac{\overrightarrow{\mathbf{p}_{\mathbf{system}}}}{\mathbf{M}}$$

i.e. net momentum of the system per unit mass of the system.

Suppose now, that the sum of external forces acting on a system of particles is zero.

$$\overrightarrow{F_{ext}} = \frac{d\,\vec{p}}{dt} = 0$$

$$\overrightarrow{\mathbf{p}} = \mathbf{constant.}$$

Thus, when the **total external force acting on a system of particles is zero**, the **total linear momentum of the system is constant.**

This is the law of conservation of the total linear momentum of a system of particles. From above equation, we can conclude that **when the total external force on the system is zero, the velocity of the centre of mass remains constant.** **(We assume throughout the discussion on systems of particles in this unit that the total mass of the system remains constant.)**

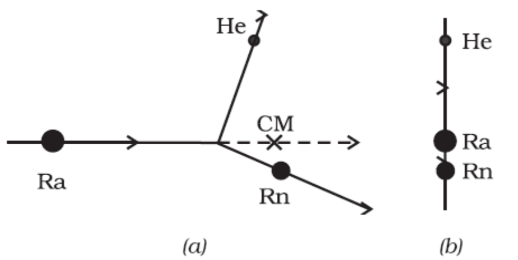
An interesting result can be drawn from equation $\vec{p} = \text{constant}$, when no external force is acting on a given system of particles or bodies with centre of mass at rest i.e. $\overrightarrow{\mathbf{v}_{\mathbf{CM}}} = \mathbf{0}$.

Then, as the velocity of centre of mass under theses condition will remain unchanged hence it will still remain at rest. In simple words the location of centre of mass will remain at the same point.

The constituents of a system due to the internal forces i.e. the forces exerted by the particles on one another, may have complicated trajectories. Yet, if the total external force acting on the system is zero, the centre of mass moves with a constant velocity, i.e., moves uniformly in a straight line like a free particle.

EXAMPLE

Let us consider the radioactive decay of a moving unstable particle, like the nucleus of radium. **A radium nucleus disintegrates into a nucleus of radon and an alpha particle**. The forces leading to the decay are internal to the system and the external forces on the system are negligible. So the total linear momentum of the system is the same before and after decay. The two particles produced in the decay, the radon nucleus and the alpha particle, move in different directions in such a way that their centre of mass moves along the same path along which the original decaying radium nucleus was moving as shown in the figure (a) given below.



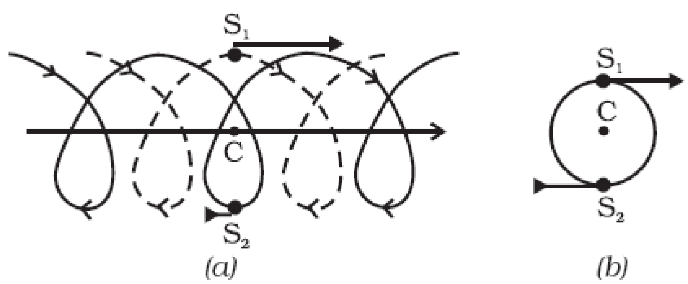
A heavy nucleus radium (Ra) splits into a lighter nucleus radon (Rn) and an alpha particle (nucleus of helium atom). The CM of the system is in uniform motion,
The same spliting of the heavy nucleus radium (Ra) with the centre of mass at rest. The two product particles fly back to back.

If we observe the decay from the frame of reference in which the centre of mass is at rest, the motion of the particles involved in the decay looks particularly simple; the product particles move back to back with their centre of mass remaining at rest as shown in above figure (b).

In many problems on the system of particles as in the above radioactive decay problem, it is convenient to work in the **centre of mass frame** or **C – frame of reference** rather than in the laboratory frame of reference. It is also known as **zero momentum reference frames.**

In astronomy, binary (double) stars is a common occurrence. If there are no external forces, the centre of mass of a double star moves like a free particle and the two stars revolve around their centre of mass as shown in figure (a) given below. **The trajectories of the two stars of equal mass are also shown in the figure; they look complicated. If we go to the centre of mass frame, then we find that there the two stars are moving in a circle, about the centre of mass, which is at rest. It is interesting to note that the position of the stars have to be diametrically opposite to each other [Figure (b)].**

Thus in our frame of reference, the trajectories of the stars are a combination of (i) uniform motion in a straight-line of the centre of mass and (ii) circular orbits of the stars about the centre of mass.



As can be seen from the two examples, separating the motion of different parts of a system into motion of the centre of mass and motion about the centre of mass is a very useful technique that helps in understanding the motion of the system.

EXAMPLE

A 2 kg particle has velocity of $2\,\hat{i} - \hat{j}$ m/s and a 3 kg particle has a velocity of $\hat{i} + 6\hat{j}$ m/s. find (i) velocity of the centre of mass and (ii) the net momentum of the system.

SOLUTION

$m_1 = 2\text{kg}$ and $m_2 = 3\text{kg}$

$\overrightarrow{\mathbf{v_1}} = 2\,\hat{i} - \hat{j}$ m/s and $\overrightarrow{\mathbf{v_2}} = \hat{i} + 6\hat{j}$ m/s

(i)

$$\overrightarrow{v_{CM}} = \frac{m_1\overrightarrow{v_1} + m_2\overrightarrow{v_2}}{m_1 + m_2}$$

$$\overrightarrow{v_{CM}} = \frac{2 \times (2\,\hat{i} - \hat{j}) + 3 \times (\hat{i} + 6\hat{j})}{2 + 3}$$

$\overrightarrow{v_{CM}} = 1.4\,\hat{i} + 3.2\,\hat{j}$ m/s

(ii) $\vec{p} = (m_1 + m_2)\,\overrightarrow{v_{CM}}$

$$\vec{p} = (2 + 3)(1.4\,\hat{i} + 3.2\,\hat{j})\ \text{kg}\frac{\text{m}}{\text{s}}$$

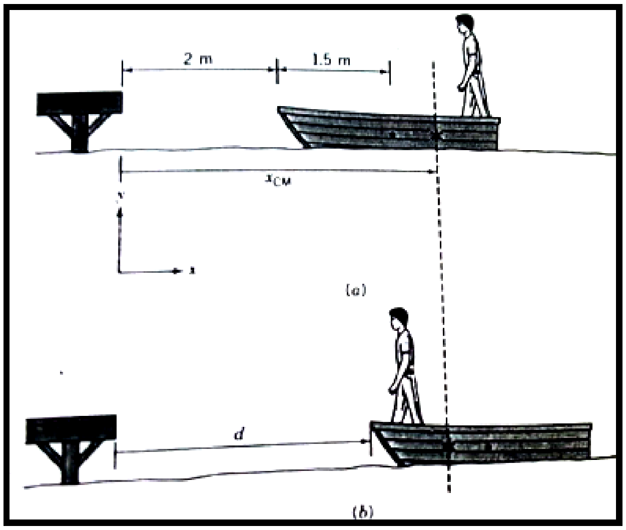
$$\vec{p} = 7\,\hat{\imath} + 16\hat{\jmath}\ \mathrm{kg}\frac{\mathrm{m}}{\mathrm{s}}$$

EXAMPLE

A man of mass 60 kg is at the rear end of a stationary boat of mass 40 kg and length 3m, which can move freely on the water. The front of the boat is 2m from the shore. What happens when the man walks to the front end of the boat? Consider the mass distribution of boat to be uniform.

SOLUTION

Let the initial position of the centre of mass of the system be given as $\mathbf{x_{CM}}$.



As the system (boat and man) is initially at rest and no external force acts on the system, therefore the position of centre of mass of the system will remain same. The boat has uniform distribution of mass hence its centre of mass will be taken at its centre.

Let all the distances are measured from the shore, then

mass of man $m_1 =$ 60 kg and mass of boat $m_2 = 40$ kg

$$\text{position of man } x_1 = 2 + 3 = 5\ \mathrm{m}$$

$$\text{position of centre of mass of boat } x_2 = 2 + 1.5 = 3.5\ \mathrm{m}$$

The initial position of centre of mass of the system of boat and man will be given as

$$x_{CM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

$$x_{CM} = \frac{60 \times 5 + 40 \times 3.5}{60+40}\,\mathrm{m} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ (1)$$

$$x_{CM} = 4.4 \text{ m}$$

After the man moves to the front end of the boat, let the front end be at a distance of d from the shore.

$$\text{new position of man } x_1 = d \text{ m}$$

$$\text{new position of centre of mass of boat } x_2 = d + 1.5 \text{ m}$$

The final position of centre of mass of the system of boat and man will be given as

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$x_{CM} = \frac{60 \times d + 40 \times (d + 1.5)}{60 + 40} \text{ m} \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

From equation (1) and (2)

$$\frac{60 \times d + 40 \times (d + 1.5)}{60 + 40} = \frac{60 \times 5 + 40 \times 3.5}{60 + 40}$$

$$d = 3.8 \text{ m}$$

NOTE: **Here the man has moved towards the shore but the boat moved away from the shore. The negative displacement of the man is accompanied by the positive displacement of the boat such that the centre of the system remains fixed.**

EXAMPLE

A 75 kg girl sits at the rear end of a wagon of mass 25 kg and length 4 m, which moves initially at 4 m/s over frictionless rails. At t = 0, she walks at 2 m/s relative to the wagon and then sits down at the front end. During the period of she was walking, find the displacements of the

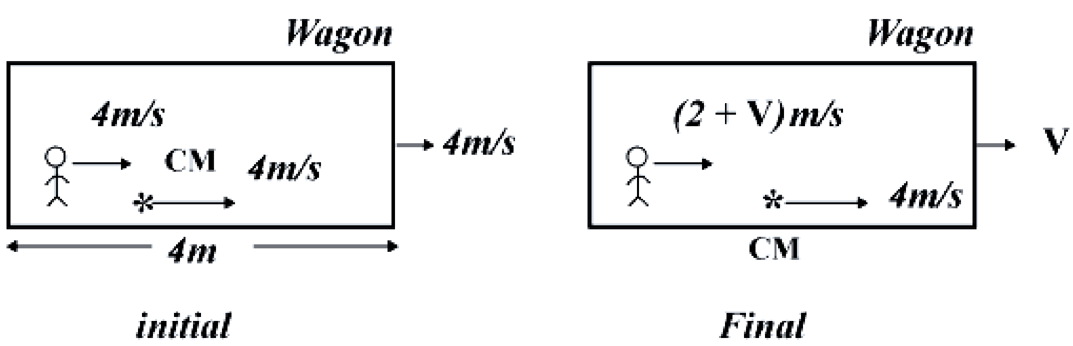
(i) **wagon**

(ii) **girl**

(iii) **centre of mass.**

SOLUTION

Initially the girl, wagon and the centre of mass has same velocity i.e. 4 m/s. as there is no external force acting on the system (wagon and girl) along the direction of motion of the wagon therefore momentum of the system will remain conserved. Also the velocity of centre mass will remain same i.e. 4m/s.



Considering the velocity of every object with respect to the ground, then initial velocity of centre of mass

$v_{CM} = 4$ m/s ………… (1)

When the girl starts walking inside the wagon ($|\overrightarrow{v_{GW}}| = 2\ \text{m/s}$), let the wagon starts moving with velocity ($\overrightarrow{v_W} = \vec{v}$) with respect to the ground, then velocity of woman with respect to ground will be

$$\overrightarrow{V_G} = \overrightarrow{v_{GW}} + \overrightarrow{v_W}$$

$V_G = 2 + V$ m/s

The velocity of the centre of mass after girl starts walking

$$\overrightarrow{v_{CM}} = \frac{m_1\overrightarrow{v_1} + m_2\overrightarrow{v_2}}{m_1 + m_2}$$

$$v_{CM} = \frac{75 \times (2+V) + 25 \times v}{75+25} \text{.................} \qquad (2)$$

From equation (1) and (2), we can write

$$\frac{75 \times (2+V) + 25 \times v}{75+25} = 4$$

Velocity of wagon $\quad v = 2.5\ m/s$

The velocity of girl $V_G = 2 + V = 4.5$ m/s

The girl moves with a velocity of 2m/s with respect to the wagon of length 4m, hence time taken by girl to go from rear end to front end of the wagon is

$$t = \frac{\text{length of wagon}}{|\overrightarrow{v_{GW}}|} = \frac{4}{2} = 2\ s$$

During this time the displacement of

- **(i)** **wagon is** $\Delta x = v\,t = 2.5 \times 2 = 5m$
- **(ii)** **girl is** $\Delta x = V_G\,t = 4.5 \times 2 = 9m$
- **(iii)** **centre of mass is** $\Delta x = V_{CM}\,t = 4 \times 2 = 8m$

7. KINETIC ENERGY OF A SYSTEM OF PARTICLES

We can now show that the kinetic energy of a system of particles can be in general can be divided into term:

- (i) the kinetic energy of the centre of mass
- (ii) the kinetic energy of the particles of the system with respect to the centre of the system

Let the position $\vec{r_i}$ of the i - th particle with respect to the fixed origin O is

$$\vec{r_i} = \overrightarrow{r_{CM}} + \overrightarrow{r_i'} \text{ ----- (a)}$$

Here $\overrightarrow{\mathbf{r}_{\mathbf{CM}}}$ is the position of the centre of mass of the system and $\overrightarrow{\mathbf{r}_i'}$ is the position of the particle with respect to the centre of mass. Taking time derivative of the above equation (a) we can find the velocity of the given particle

$$\overrightarrow{\mathbf{v}_i} = \overrightarrow{\mathbf{v}_{\mathbf{CM}}} + \overrightarrow{\mathbf{v}_i'}$$

The kinetic energy of the i - th particle with respect to the fixed origin O is, hence

$$\mathrm{KE_i} = \frac{1}{2}\mathrm{m_i}\overrightarrow{\mathbf{v}_i} \, . \overrightarrow{\mathbf{v}_i}$$

$$\mathrm{KE_i} = \frac{1}{2}\mathrm{m_i}\left(\overrightarrow{\mathbf{v}_{\mathbf{CM}}} + \overrightarrow{\mathbf{v}_i'}\right).\left(\overrightarrow{\mathbf{v}_{\mathbf{CM}}} + \overrightarrow{\mathbf{v}_i'}\right)$$

$$\mathrm{KE_i} = \frac{1}{2}\mathrm{m_i}\left(\mathbf{v}_{\mathbf{CM}}^2 + \mathbf{v}_i'^2 + \mathbf{2}\,\overrightarrow{\mathbf{v}_{\mathbf{CM}}} \, . \overrightarrow{\mathbf{v}_i'}\right)$$

The total kinetic energy of the system can be can be written as

$$\mathrm{KE} = \sum \mathrm{KE_i}$$

$$\mathrm{KE} = \frac{1}{2}\left(\sum \mathrm{m_i}\right)\mathbf{v}_{\mathbf{CM}}^2 + \sum \frac{1}{2}\mathrm{m_i}\,\mathbf{v}_i'^2 + \overrightarrow{\mathbf{v}_{\mathbf{CM}}}.\left(\sum \mathbf{m_i}\,\overrightarrow{\mathbf{v}_i'}\right)$$

The first term, $\frac{1}{2}(\sum \mathrm{m_i})\mathbf{v}_{\mathbf{CM}}^2 = \frac{1}{2}\,\mathrm{M}\,\mathrm{v}_{\mathrm{CM}}^2$, is the product of sum of all the masses and square of speed of centre of mass. While the second term, $\sum \frac{1}{2}\mathrm{m_i}\,\mathbf{v}_i'^2$, is the kinetic energy of the system of the particles with respect to the centre of mass of the system.

It can be also termed as kinetic energy of the system in centre of mass frame. The third term has quantity, $\sum \mathrm{m_i}\,\overrightarrow{v_i'}$, which is the total momentum of the system relative to the centre of mass. As we know the total momentum of a system is zero when measured with respect to the centre of mass. Therefore the last term will become zero.

Hence equation (8) can be written as

$$\mathrm{KE} = \frac{1}{2}\left(\sum \mathrm{m_i}\right)\mathbf{v}_{\mathbf{CM}}^2 + \sum \frac{1}{2}\mathrm{m_i}\,\mathbf{v}_i'^2$$

$$KE = K_{CM} + K_{rel}$$

$$K_{CM} = \text{kinetic energy of centre of mass}$$

$$K_{rel} = \text{kinetic energy of the system with respect to the centre of mass}$$

The K_{rel} may involve translation, rotation or vibration relative to the centre of mass. It can be termed as the internal energy of the system.

Consider an example of a gas, of mass M, filled cylinder kept in a moving vehicle with velocity v. We know the gas molecules move randomly inside the container. The velocity of vehicle will also be the velocity of the centre of mass of the gas system.

$$KE = K_{CM} + K_{rel}$$

Then using above relation we can write the expression of kinetic energy of the gas as:

$K_{CM} = \frac{1}{2} Mv^2$, while K_{rel} can also be termed as internal energy of the gas.

EXAMPLE

Consider two particles of masses m_1 and m_2 moving with velocities v_1 and v_2 along positive x axis. Deduce the expression of the kinetic energy of the system about its centre of mass.

SOLUTION

The velocity of the centre of mass of the given system can be given as

$$v_{CM} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

$$K_{rel} = \frac{1}{2} m_1 (v_1 - v_{CM})^2 + \frac{1}{2} m_2 (v_2 - v_{CM})^2$$

Where $(v_1 - v_{CM})$ is the velocity of m_1 w.r.t. to centre of mass and $(v_2 - v_{CM})$ is the velocity of m_2 w.r.t. to centre of mass.

Therefore

$$K_{rel} = \frac{1}{2}m_1\left(v_1 - \frac{m_1v_1 + m_2v_2}{m_1 + m_2}\right)^2 + \frac{1}{2}m_2\left(v_2 - \frac{m_1v_1 + m_2v_2}{m_1 + m_2}\right)^2$$

$$K_{rel} = \frac{1}{2}\frac{m_1(m_2)^2}{(m_1 + m_2)^2}(v_1 - v_2)^2 + \frac{1}{2}\frac{m_2(m_1)^2}{(m_1 + m_2)^2}(v_1 - v_2)^2$$

$$K_{rel} = \frac{1}{2}\frac{m_1m_2}{(m_1 + m_2)^2}(m_1 + m_2)(v_1 - v_2)^2$$

$$K_{rel} = \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}(v_1 - v_2)^2$$

$$K_{rel} = \frac{1}{2}\mu v_{rel}{}^2$$

Here $\mu = \frac{m_1m_2}{m_1+m_2}$ is referred as the **reduced mass.** While$v_{rel} = v_1 - v_2$ is the relative velocity of the two particles taken, which interestingly is independent of the frame of reference.

Reduced mass is a value of a hypothetical mass introduced to simplify the mathematical description of motion two-body system. The equations of motion of two mutually interacting bodies can be reduced to a single equation describing the motion of one body in a reference frame centered in the other body. The moving body then behaves as if its mass is reduced and given as

$$\boldsymbol{\mu = \frac{m_1m_2}{m_1 + m_2}}$$

This concept is can help in solving the problems related with two interacting objects like two blocks tied to spring, binary star system, nucleus electron system etc.

8. WORK ENERGY THEOREM FOR A SYSTEM OF PARTICLES

As we are aware of the **work energy theorem for a single particle is stated as the net work done is equal to the change in kinetic energy of the particle**

$\mathbf{W_{net} = \Delta KE}$

If a system of particles is considered, then above expression holds true for each particle. As we know internal forces cancels out in pairs according to Newton's third law of motion, but if the particles can move relative to each other or inter particle separation changes then the work done due to internal forces can't be taken as zero.

For example isolated system of blocks held against a compressed spring, block strikes another block with a spring attached etc are few examples where such a situation takes place.

The work energy theorem must include external and internal work

Therefore

$$W_{ext} + W_{int} = \Delta KE$$

Using equation:

$$KE = K_{CM} + K_{rel}$$

In the centre of mass frame or C frame of reference, we can write the above equation as

$$W_{ext} + W_{int} = \Delta K_{CM} + \Delta K_{rel}$$

As all basic interactions are conservative in nature therefore

$$W_{int} = -\Delta U_{int}$$

Where ΔU_{int} is the change in internal potential energy of the system.

Therefore, we can write

$$W_{ext} - \Delta U_{int} = \Delta K_{CM} + \Delta K_{rel}$$

$$W_{ext} = \Delta K_{CM} + \Delta K_{rel} + \Delta U_{int}$$

$$W_{ext} = \Delta K_{CM} + \Delta(K_{rel} + U_{int})$$

$$W_{ext} = \Delta K_{CM} + \Delta E_{int}$$

Where ($E_{int} = K_{rel} + U_{int}$) can be defined as the internal energy of the system which includes translation and rotational kinetic energy relative to the centre of mass and internal potential energy (spring elastic energy, gravitational energy, thermal energy, electromagnetic energy, chemical energy and nuclear energy).

Hence the equation,

$$W_{ext} = \Delta K_{CM} + \Delta E_{int}$$

tells us that work done due to external forces on a system of particles can change the translation kinetic energy of the centre of mass and internal of the system, kinetic with respect of centre of mass and potential energy.

The expression:

$$W_{ext} = \Delta K_{CM} + \Delta E_{int}$$

can lead to interesting result when the system is isolated i.e. no external force acts on a system or $W_{ext} = 0$. Then in that case:

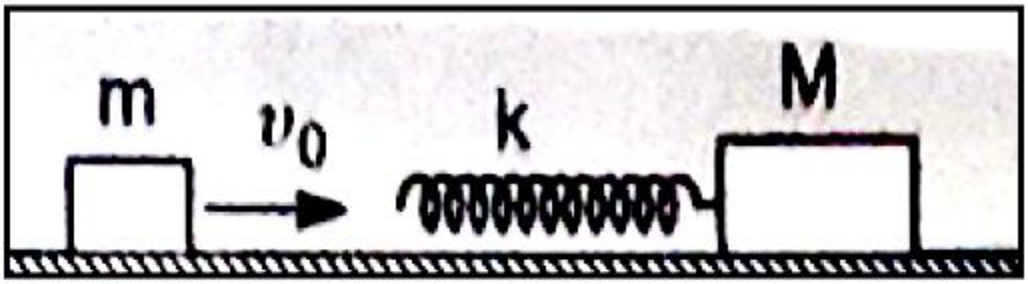
$$\Delta K_{CM} + \Delta E_{int} = 0$$

$$\Delta K_{CM} = -\Delta E_{int}$$

The system centre of mass can gain kinetic energy at the expense of loss of its internal energy or vice versa.

EXAMPLE

A block of mass m moves with velocity V_0 towards a stationary block of mass M on a smooth horizontal surface as shown in the figure. Find the maximum compression in the spring of stiffness constant k.



SOLUTION

The velocity of the centre of mass of the system will remain unaffected before and after the impact as there is no external force acting on the system. We can say the springs will compress as long as the block have relative motion with respect to each other or in the Centre of mass frame of reference the blocks are rest.

Applying work energy theorem in the Center of mass frame of reference

$W_{ext} = \Delta K_{CM} + \Delta E_{int}$……. (1)

Here $W_{ext} = 0$…………… (2)

As the velocity of the centre of mass remains same, therefore the change in kinetic energy of the centre of mass will be zero

$\Delta K_{CM} = 0$…………………. (3)

From (1), (2) and (3) we can write

$$-\Delta E_{int} = 0$$

Initial and final value of energy of the system with respect to the centre of mass will remain same

$$E_{int,initial} = E_{int,final}$$

As $E_{int} = K_{rel} + U_{int}$

As $K_{rel} = \frac{1}{2}\mu {v_{rel}}^2$ and $U_{int} = \frac{1}{2}kx^2$

Initially Spring is not stretched i.e. x = 0 and $v_{rel} = v_o - 0$

$E_{int,initial} = \frac{1}{2}\mu {v_{rel}}^2 + 0 = \frac{1}{2}\frac{m\,M}{m+M}(v_o)^2$.............. (4)

Finally, spring is compressed by, let's say x and $v_{rel} = 0$ as both blocks are at rest with respect to each other. Therefore

$E_{int,final} = 0 + \frac{1}{2}kx^2$....................................... (5)

From (4) and (5)

$$\frac{1}{2}kx^2 = \frac{1}{2}\frac{m\,M}{m+M}(v_o)^2$$

$$x = v_o\sqrt{\frac{m\,M}{m+M} \times \frac{1}{k}}$$

Therefore $x = v_o\sqrt{\frac{\mu}{k}}$ where μ is the reduced mass of the system

9. SUMMARY

- Rigid bodies are made up of a large number of particles, separation between them remains the same in time.
- Motion of rigid body can be described using the concept of center of mass

- To determine the motion of the centre of mass of a system no knowledge of internal forces of the system is required. For this purpose we need to know only the external forces on the body. Separating the motion of a system of particles as the motion of the centre of mass, (i.e., the translational motion of the system) and motion about (i.e. relative to) the centre of mass of the system is a useful technique in dynamics of a system of particles.

- Newton's Second Law for finite sized bodies (or systems of particles) is based in Newton's Second Law and also Newton's Third Law for particles.

- The centre of mass moves, as if all the mass of the system is concentrated at this point and all the external forces act at it. the acceleration of the centre of mass can be determined as

$$\overrightarrow{a_{CM}} = \frac{\overrightarrow{F_{ext}}}{M}$$

Where $\overrightarrow{F_{ext}}$ is the net external force acting on the system of particles and M is the mass of the system.

- The velocity of the centre of mass of a system of particles is given by $\overrightarrow{v_{CM}} = \frac{\overrightarrow{p_{system}}}{M}$ where $\overrightarrow{p_{system}}$ is the linear momentum of the system and M is the net mass of the system.
- The centre of mass moves, as if all the mass of the system is concentrated at this point and all the external forces act at it.
- If the total external force on the system is zero, then the total linear momentum of the system is constant.

- The kinetic energy of a system of particles KE can be given as:

$$KE = K_{CM} + K_{rel}$$

Where K_{CM} is the kinetic energy of centre of mass.

And K_{rel} = kinetic energy of the system with respect to the centre of mass

- The K_{rel} may involve translation, rotation or vibration relative to the centre of mass. It can be termed as the internal energy of the system.

For a system of particles the work energy theorem can be written as

$$W_{ext} = \Delta K_{CM} + \Delta E_{int}$$

Where $E_{int} = K_{rel} + U_{int}$

It states that work done due to external forces on a system of particles can change the translation kinetic energy of the centre of mass and internal of the system, kinetic with respect of centre of mass and potential energy.

- The internal energy of the system includes translation and rotational kinetic energy relative to the centre of mass and internal potential energy (spring elastic energy, gravitational energy, thermal energy, electromagnetic energy, chemical energy and nuclear energy).